Modelling, Functions and Estimation. A Pizza Problem.

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As part of the Bowland Trust project to produce teaching materials to support applications of mathematics, King's College, London developed a project in which students watched pizzas cool. The temperature of the pizzas is measured over time using data logging apparatus.

The pizza shop owner has hired mathematical consultants to support them in maximising their market and hence their profit. Pizzas need to be developed fresh and hot. Many issues need to be addressed in this project, but two key questions emerge. How far can my deliveries reach and the pizzas remain sufficiently hot? Does it matter how I package them? This requires two experiments, firstly to determine the least acceptable temperature and secondly to determine the time taken to reach this temperature with different packaging. The first is simple and requires a volunteer prepared to eat small pieces of pizza as it cools. Generally, the volunteer is not hard to find! It is the second experiment which throws up a lot of interesting mathematics.

It is important to recognise that there is no pizza shop, nor will our report actually affect one. Our students are principally motivated by a desire to get on in their maths. Frequently scenarios are presented in class as problems being solved as if they were in the 'real world' that is, motivated by the requirements of the supposed scenario. Paul Dowling refers to this as the myth of reference. (Dowling, 1998). Clearly, pizza shop owners are unlikely to employ consultants with data logging apparatus and computer algebra systems for advise on their business plan. However, the scenario brings a welcome sense of fun, plus the warm smell of pizza to the mathematics classroom. This recognition is important. Details of the mathematical development could be argued as unnecessary for scenario and the mathematical development would be restricted. We need to retain our perspective on our real purpose, being to develop the maths. We do nonetheless have a setting which motivates the desire for quality outside our area of expertise. The two experiments require some serious scientific consideration. Indeed when we have presented this idea to adult teacher professionals, the focus has often been on critiquing the quality of the experimental set-up. Students in our trial schools did not. However, it clearly provides an opportunity to collaborate with science departments, who would wish to discuss the experimental design and improve it. Equally, an initial discussion about the maximising the profit of the pizza business throws up many issues (e.g. total profitability, cost/benefit of additional employees, food quality issues, etc.) that are evidently beyond the scope of the maths teacher. Instead of trying to 'deal' with them in a pretend fashion, these could be worked up seriously in the business studies department, where this expertise resides.

What do students see when they look at a graph? Activities involving story graphs are frequently designed to help the student visualise the change in one variable dependent on another. However, constructing the story is difficult and the simplifications in the graphical representation often strain credulity. The strange stories of children walking with constant speed being a case in point. I have frequently used real time distance logging apparatus with experienced teachers and have been struck by how often they walk towards it when the distance/time graph they are supposed to be tracking goes up or ask me where they should start when the graph clearly shows the distance at time zero. (See, Teachers TV, 2006). That there is a complicated link, often weak, between the scenario and the graph seems clear. Having, the possibility to collect the data in real time as a clearly known process is unfolding in front of the learners eyes (and nose!) provides a powerful link.

Jeremy Rochele has developed SimCalc, a software simulator which produces cartoon images of motion activities (for example characters running at different speeds) together with a graph and table of values. This provides an example of multiple representations (see Ainsworth and

VanLabeke, 2004) which is a key design precept of all graphing calculator technology. Rochele et al presents a outcomes from control and tr4eatment groups involved in using the SimCalc to study "the mathematics of change and variation". They found "Although, on average, Treatment and Control group students progressed equally well on simple mathematics, the Treatment group gained more on complex mathematics. For example, at post test, Treatment students were more likely to use the correct idea of "parallel slope as same speed," whereas Control students were more likely to have the misconception "intersection as same speed." (Rochelle et al. 2007). Test items examined exactly the misconceptions that SimCalc is designed to address, so it would be interesting to compare the quality of the input received by the control group. Nonetheless, this seems to have been effective in generating a felt link between motion and graph. In our case the measured change is happening in reality (rather than virtual reality) which may perhaps create a stronger link, although this remains to be examined.

It is necessary to be clear about our purpose here. We intend to watch the change in temperature of a pizza over time, in order to find out how long it takes to reach a certain value. (In our experiments we found 48° to be the least acceptable temperature). Now this could take a long time, longer than we could reasonably continue measuring for (certainly in an ordinary lesson). So, we will see if we can find a rule for the rate of cooling, that will enable us to predict how long it will take. The predicting aspect requires the setting up and critiquing of a mathematical model, yet seems sensible enough in the context of the scenario. It is routine in classrooms to ask students to estimate. However students need to have the opportunity to develop their skills in estimation and critically reflect on the how they estimate the future temperature.

In the classroom we tried two different models. One featured the teacher controlling the experiment using one microwave oven placed at the front of the classroom with one probe and computer set-up. The second featured 6 groups of students taking turns to use one of two microwave ovens at either end of the room, each group having its own probe and computer. The data logging equipment produces a real time graph which shows how the temperature is decreasing with time. It also shows the temperature on screen. We produced a worksheet in which students are asked to predict the initial temperature (actually the point at which the temperature starts decreasing, to take account of the probe heating up). Next to each prediction is a space to state the basis on which the prediction was made. Initially this will be due to external factors (guess, how hot ovens get, etc.). As soon as the pizza comes out of the oven it was placed inside one of the packaging types and the probe inserted. (Mini deep pan pizzas were used to ensure that the temperature of the topping was being measured, rather than the base). Students had already made their prediction for the peak temperature (i.e. time zero), so the data logger was set running. Immediately students are asked to predict the temperature at the end of the first minute. The experiment is timed (we used a volunteer timekeeper to shout out 5-4-3-2-1 at the end of each minute). At the end of the first minute, the actual temperature is logged on the worksheet and a prediction is made for the end of the second minute. Now, some students start to look at the rate of cooling as an indicator to support their estimate for the end of the second minute. Also, students are asked to estimate the temperature at the end of the fifth minute. The experiment continues in this way for 10 minutes, each time students make future estimates. After five minutes they estimate for end of the sixth minute and the end of the tenth minute. By now, students are taking close account of the rate of change and using this to make better and better estimates for each successive minute. There is an are of quiet competitiveness and satisfaction when estimates are close or even perfect. (Reading are taken to one decimal place). After ten minutes, they estimate for 30 minutes, 120 minutes and 24 hours. Again the requirement to explain the basis for the estimation is emphasised. This last part requires students to share their mechanisms for estimation and discuss how they expect the temperature to change after the experiment has ended, i.e. into the never-to-be-known. It was gratifying that students happily watched a pizza warming in a microwave oven for two minutes then watched it cooling for 10 with rapt attention!

At the end of the experiment it is clear that the pizza is nowhere near down to the 48° minimum, so we need some way to work out when it will reach that temperature. The stage is now set for the key conversation: on what basis were the estimates made. Typically the cooling graph looks very linear. When asked how they estimated most considered answers were along the lines of "it was going down about 2.3° a minute" or sometimes "for the first five minutes, it was going down about 2.4° a minute and then for the next five minutes, about 2.1° a minute", Both clear statements of linearity. Depending on the available equipment, the students either drew a graph by hand from the data on the worksheet, or had access to a dynamic graph within the data collection software. By tradition, a line of best fit seems an obvious thing to make, so the possibility of setting up a model comes out naturally. Starting with the simpler suggestion of "going down 0.8° a minute" we can ask, so what was the starting temperature? The software allows us to enter a function, to fit the data. So we start at the peak temperature (in the example it is 84°). So starting with f1(x) = 84 makes sense. Then it went down by 2.3° per minute so we make it f1(x)=84−2.3x. The set up makes this look very natural. But then when we hit return to draw our best fit line something is clearly wrong. It requires very little prompting to see that the 2.3° was per minute, but the data was gathered per second. So we can edit the function to show f1(x)=84−(2.3/60)x. Happily the software gives an immediate response, so testing different theories for accommodating the minutes to second conversion can be done quickly by test and check. This feature keeps the discussion on track and avoids being sidelined by tricky numeracy issues. These can be returned to later. Often the very beginning of the experiment has an uneven cooling rate, so a little 'tweaking' of the model needs doing. Having seen the construction of the model, students feel in control of the coefficients. The can move it up or down a bit by changing the 'starts at' value and change the steepness by varying the 'per minute' value. They are very impressed by their capacity to make a near perfect fit.

Looking at the graph of our best fit function, we can see roughly when the temperature will be down at 48°, this may require extending the axes. However, immediately the estimating power of the function becomes clear. Students who hand drew their graphs immediately see the flexibility of the software. Doing this by eye is very useful as it reinforces the power of the function. This we can now move to as we know that this functions fits our data very well and we want to know when the value of this function is 48. That is 84−(2.3/60)x = 48. Immediately students recognise that this is an equation. Their knowledge of how to solve it can now be brought to bear. Powerfully the software includes a computer algebra system (CAS). Here we can state the equation. Then work on it in whichever way students suggest. Sensible and not so sensible suggestions can be tested and their outcome considered. That there are many routes to solution is very empowering here. In the CAS we simply type the equation and enter it. Then take simply state what we wish to do to both sides. (CAS rightly cannot accept a fraction of 2.3/60 and so writes it correctly as 23/600, generating another key intervention). In the example, we took away 48, then took away 36, then multiplied by 600, then divided by -23. (A route suggested by a student). The CAS shows the result of an operation applied to the whole equation. This is quite a striking notion and has subtle advantages over the 'both sides' argument. That we have taken away 48 from the whole equation is more resonant with ideas of equivalence between statements. This does give us the potential for an interesting discussion later. This is about 15 and a half minutes. For more complicated functions we may not (yet) have the tools to find a solution, so it is useful to demonstrate the solve function in CAS. We simply define a function f(x):= 84−(2.3/60)x. (Note that we use := i.e. is defined as. The different uses of the equals sign are frequently glossed in classrooms and remain a key source of algebraic confusion for students). Here we are forced to recognise the difference between the function definition and the equation which we solved). It is good at this stage to test a few values (e.g. f(0), f(60) etc.) to reinforce students' confidence in the function. We can now use the CAS command **solve(f(x)=48,x)** and find the same answer as we found using the traditional method. This will become very powerful as more sophisticated functions are found to be necessary.

There is considerable debate on the merits of computer algebra systems with keen advocates promoting their use against a concern for the clear requirement to effect substantial change in assessment systems predicated on routine solutions which CAS can perform for the user. (e.g. Bohm et al, 2004) In this context the CAS is being used to support and sustain the mathematical narrative. Discussing possible curriculum change in Australia, Driver suggests that CAS "... can be used to "do the messy algebra". By allowing a student to focus on the selection of a problem solving strategy or appropriate procedure rather than the application of the strategy or procedure, and student can develop their higher-order thinking skills". (Driver, 2008). This is our purpose here. Beyond the linear case, the graphical transposition and equation solving would be difficult and would certainly get in the way of the narrative flow. Even the linear case requires effective routine facility, which if not secure will change the focus of the narrative. It does nonetheless engage the student with the need for this facility and more clearly motivate its development at a later point.

Returning to the narrative, we now have function which fits our data, so we can test its predictive capabilities. At the end of the experiment, students estimated the temperature after longer periods. In discussion, the basis on which these estimates were made changed from the short term mechanism of the linear decrease. After 30 minutes and certainly after 120 minutes most students are suspecting that the rate of decrease will have slowed. Some students suspect that after 24 hours the pizza will only have reached room temperature. So, we can test these in the function. Student's commit their expectation of the outcome to paper, first. Neatly, the CAS can deal with an input like f(24*60*60) to test the 24 hour figure. The outcomes for 30 minutes, 120 minutes and 24 hours respectively, provide an increasing surprise and realisation that something is wrong. Going back to the graph and extending the horizontal axis progressively, provides a visual confirmation. Students are able to interpret the graph now that they have identified the relationship between the downward graph and the cooling pizza. They are, of course, very well aware that, left to their own devices overnight, pizzas do not continue cooling, freezing and ultimately exceeding absolute zero! So, they are well oriented to finding a function that fits the data, but does not continue to decrease in this way.

Lesh et al developed modeling activities for a range of groups from middle school students to graduate students, they found that, "Few students who worked on this version went through more than half of a modeling cycle; and, almost none persevered to the point where they could make even an educated guess regarding predicted gains. After producing "first-draft answers," these students did not feel any need to produce second- or third-draft answers". (Lesh et al, 2008). They went on to develop their activities, but explicitly to present a second activity requiring a different analysis. The pizza scenario has the advantage that the first (linear) model is overwhelmingly favourite (with student and teacher participants) and the critique of this model is clearly grounded in participants existing knowledge of the scenario. However, the insight gained by reflecting on the basis for the estimates sets up the natural concern that the linear model isn't quite right. Hence, the second (and third) iteration appears as necessary development.

Hence, students are now free to explore different functions. The key feature they have seen is the ability to control the shape and position of the graph by varying the coefficients in the function. A base function such as $f(x)=x^2$ The graph can then be dragged in all directions and scaled by dragging out. This shows the function in the completed square form, which is a very powerful form for transposing the graph. An added bonus here is developing the recognition that different forms of an equation are powerful in different ways. The completed square form often only being seen as a long winded way of solving an equation. $f(x)=x^2$ is the most common next function for students as they are often aware of it's existence and perhaps conscious the shape of its graph, meeting the problems with the linear function. Once in control of the coefficients, students find a quadratic which accurately fits the data. They can then use CAS tools from before to solve the equation $f(x)=48$ and test the accuracy over the longer term. It becomes clear with this analysis that the function falls down in the longer term because it seems to suggest that pizzas will cool to a minimum and then start heating up again, becoming very hot indeed by the following day. Once again this does not accord with the students' common sense notion of how pizzas actually behave. Now, they have a very strongly formed mental image of the shape of the graph of the function they are looking for. In out lessons we restricted exploration to a palette of possibilities consisting of linear, quadratic, reciprocal and exponential function. However, the software can cope with other interesting functions, which we have tried out in teacher sessions. Notably, piecewise linear functions. These naturally accord with the descriptions of the variation students suggested i.e. A certain rate of decrease over a certain range, followed by a different rate of decrease over the next part of the range. With a final linear function of f(x)=room temperature after a certain period, this can be an excellent model. Clearly, a well constructed exponential will also provide an excellent model.

It is clear by this stage that we have far exceeded our requirement to find the time for the pizza to cool to 48°. However, we have developed our skills in finding and evaluating models for the cooling function. There is now a high degree of confidence that we can find the cooling time effectively in different situations. The experiment can now be repeated with different types of packaging. The the consultants are ready to report to the pizza shop owner. With the added calculation of the distance possible given a known average speed from the delivery vehicle, the problem becomes one which geographers and business studies specialists may be better able to support.

I have provided a detailed description of a classroom narrative. The structuring of the narrative is built around a mathematical modelling activity, which is itself couched in a realistic but fictitious scenario. The structure is carefully designed to continually provide the rationale for further development of the theory. There are two principle mathematical outcomes: (i) an engagement with the process of mathematical modelling per se. and (ii) functions and there graphs.

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